MEMBERSHIP FUNCTION FORMULATION
METHODS FOR FUZZY LOGIC SYSTEMS: A COMPREHENSIVE REVIEW

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ABSTRACT: Fuzzy logic has a widespread application in industry, some of the common applications of fuzzy logic lie in the field of automobiles, consumer electronics, image processing, machine learning, and non-linear control systems to name a few. Due to this fuzzy logic has seen a lot of advancements since its introduction by Zadeh. Designing a proficient fuzzy logic control system is governed by several design parameters which include: controller architecture, fuzzification method, membership function formulation, rule base acquired from expert knowledge, inference engine, and defuzzification method. With an objective to help the first time users of the fuzzy logic system, this paper aims to provide a detailed analysis of the existing techniques utilized by researchers and industrial practitioners to design an optimum fuzzy logic system.

KEYWORDS: Fuzzy logic, Membership function, Optimization, fuzzy entropy.

I. INTRODUCTION

Designing a proficient fuzzy logic system is governed by several design parameters which include: controller architecture, fuzzification method, membership function formulation, rule base, inference engine, and defuzzification method. As the notion of fuzzy logic is based on uncertainty, an idea of having an empirical formula to determine membership function defies with the generalized applicability of the fuzzy logic system. Optimization of membership function has always been a field of research in fuzzy logic systems. This paper provides a detailed analysis of the existing techniques utilized by researchers and practitioners to design a fuzzy logic system. This paper presents an overview of the development in the methods of membership function formulation and their applications. With the increase in complexity of a system, the proficiency to make accurate statements about its performance decreases until a threshold is reached beyond which these attributes become almost mutually exclusive. As the system becomes more complex, uncertainty in the system arises in the form of ambiguity, impression, fuzziness, discord, vagueness [1]. In some complex systems, the available numerical data is less and mostly imprecise, and ambiguous data is available, in such a case fuzzy reasoning may be useful in understanding the behavior of a system by incorporating between input and output [2-3]. FLC has found its use in several practical applications [2, 4-6]. FLC based system is a nonlinear mapping between input and output variables i.e. they are utilized for inferring complex nonlinear systems. The nonlinear mapping property of FLC is predominant when the physical mechanism to be controlled is inherently nonlinear (primarily due to parametric uncertainty or time-variant systems) [7]. The competency of FLC in control performance for these systems has been established by multiple sources [8-10]. Control system design involves the analysis and formulation of the dynamic behavior of the system to be controlled, following which a control algorithm with an objective to attain anticipated predefined control objectives is developed, and most of the real-world systems are primarily nonlinear in nature [8]. The working principle of FLC is different from conventional stochastic models. FLCs do not make presumptions regarding the process based on the probability distribution model. This distinction of FLC makes them particularly applicable to nonstationary systems. Fuzzy logic systems can approximate any real continuous function to a compact set, making FLS suitable for approximating a dynamic system to any degree of accuracy [11]. In [12] author presented an FLC for real-time control of a magnetic levitation system. In [13] author proposed tuning of a fuzzy type PID controller by using the particle swarm optimization method. The proposed algorithm was implemented to control an electrical DC drive system; simulation and experimental results indicated better efficiency and robustness of the proposed controller. In [6] author proposed optimization of fuzzy system using cross-mutated operation in particle swarm optimization (PSO) technique. The robustness of the algorithm is tested on; economic load dispatch system and self-provisioning systems for communication network services. The results indicate a more efficient system and better robustness as compared to the hybrid PSO technique. Numerous control strategies have been applied to control IP; FLC being one of them. Some of the recent works include [14] which uses an adaptive FLC based on
feedback linearization utilizing a fuzzy disturbance observer to ensure asymptotic stability. In [15] author proposed an optimized FLC for control of an IP which is optimized by minimizing the mean square error based objective function. Performance comparison shows improved stability for the system. In [16] author evaluated the applicability of FLC to control non-linear by treating the system as a black box system; without considering system dynamics. The results indicate the effectiveness of FLC to control non-linear IP.

The paper presents a systematic and detailed study for fuzzy membership function formulation. Section 2 describes the basics of the membership function. Section 3 describes various methods and algorithms used for the estimation of membership function along with their applications and performance estimation. Section 4 discusses fuzzy entropy and methods of membership function estimation based on entropy calculations. Section 5 concludes the research and provides the reader with a research gap.

II. MEMBERSHIP FUNCTION

Fuzzy sets can be defined as a set with a vague (ambiguous) boundary as compared to a crisp boundary of classical sets. For a discrete universe of discourse (finite), \( x \) \( A \), and is depicted as follows [17]:

\[
A = \left\{ \mu_A(x_1) + \mu_A(x_2) + \ldots \right\} = \sum_{i=1}^{n} \mu_A(x_i)
\]  

(1)

here, \( \mu_A(x_i) \) is a value between 0 and 1.

For the continuous universe of discourse \( x \), an FS is denoted by:

\[
A = \left\{ \int_{X} \mu_A(x) \right\}  
\]

(2)

In equations (1) and (2), the horizontal is used as a delimiter and does not specify quotient. The numerator expression depicts the membership value of set \( A \) associated to the element \( x \) in the universe which is depicted in the denominator. In equation (1), the summation symbol denotes the collection of each element and does not signify algebraic summation. Similarly in equation (2) the integral sign depicts a continuous function-theoretic aggregation operator for continuous variables and does not signify algebraic integral operator.

Figure 1 (a) Crisp set vs. FS (b) A typical FS with set parameters [18]

Figure 1 (a) illustrates a fuzzy set where a comparison between classical (crisp) set and an FS is depicted. In crisp sets there is no ambiguity of membership grade for an element and its membership value is either 0 (not an element of the set) or 1 (an element which belongs to the set), whereas in an FS membership grade may lie anywhere between 0 and 1 (both inclusive). Here 1 is the highest membership value and 0 is the lowest membership value [18], [19]. Figure 1 (b) depicts a normal convex fuzzy set
According to standard definitions, the parameters for MF are defined as [18]:

1. **Support** – it is defined as the region of the universe having a non-zero membership in a given fuzzy set \( \tilde{A} \) \( \mu_{\tilde{A}}(x) > 0 \).

2. **Core** – it is defined as the region of the universe which has a membership value of 1 in a given fuzzy set \( A \) \( \mu_A(x) = 1 \).

3. **Boundary** – it is defined as the region of universe which has a membership value between 0 and 1 in a given fuzzy set \( \tilde{A} \) \( 0 < \mu_{\tilde{A}}(x) < 1 \).

4. **Normal FS** – it is a membership function having a membership value of 1 for at least one element in the universe.

5. **Convex FS** – it is a membership function having membership values: (a) monotonically increasing, or monotonically decreasing, or (b) whose membership values are monotonically increasing then decreasing with increasing values for elements in the universe.

### 2.1 Types of Membership Function

Some common mathematical functions which govern the shape a MF are enumerated below [18, 20-23]:

**Triangular MF** – one of the simplest and most commonly used MFs for designing FLS. The fundamental attributes of triangular MFs which differentiate it from other MFs are:

- Here the boundary varies linearly from highest to lowest membership grade
- There’s only 1 discrete element having the highest membership grade.

A triangular MF is characterized by the following equation:

\[
\mu_A(x) = \begin{cases} 
0, & x \leq a \\
\frac{(x-a)}{(b-a)}, & a < x \leq b \\
\frac{(x-c)}{(b-c)}, & b < x \leq c \\
0, & x \geq c 
\end{cases}
\] (3)

Here the range “a to c” depicts the support of the fuzzy set and “b” is a unique point in the range having the highest membership function value.

**Trapezoidal MF** – Like triangular, the trapezoidal also has a linear boundary for FS; however, a trapezoidal MF is characterized by a range of elements having maximum membership grade. Trapezoidal MF is represented by:

\[
\mu_A(x) = \begin{cases} 
0, & x < a \\
\frac{(x-a)}{(b-a)}, & a \leq x < b \\
1, & b \leq x < c \\
\frac{(d-x)}{(d-c)}, & c \leq x < d \\
0, & x \geq d 
\end{cases}
\] (4)
Gaussian MF – Gaussian MFs are characterized by non-linear boundaries for MF variation and are defined as follows:

\[
\mu_A(x) = e^{-\frac{(x-c)^2}{2\sigma^2}}
\]  \hspace{1cm} (5)

Here, \(\sigma\) denotes the standard deviation, and \(c\) is considered as a curve-fitting constant, governing the shape of the MF.

Probability Density Function – This FS is obtained from the probability distribution of a variable. The MF function is given by:

\[
\mu(x) = 1 - \frac{x}{a}, \hspace{0.5cm} x \in [0,a]
\]  \hspace{1cm} (6)

For a nonlinear function, the linearized form is represented as

\[
\mu(x) = \alpha(1 - e^{(b-x)(b-a)}), \hspace{0.5cm} x \in [a, b]
\]  \hspace{1cm} (7)
S function – These FS are characterized by non-linear boundaries for MF variation. These are commonly used for analytical approximation, and are depicted as:

\[ S(x) = \begin{cases} 
0, & x \leq a, \\
\frac{2}{(c-a)^2} (x-c)^2, & a \leq x \leq b, \\
1 - \frac{2}{(c-a)^2} (x-c)^2, & b \leq x \leq c, \\
1, & x \geq c,
\end{cases} \tag{8} \]

here a, b, c are parameters used for the shaping of the curve, and b is the mid-point.

III. DEVELOPMENT OF MEMBERSHIP FUNCTION

These developments can be categorized as:
1. Theoretical developments: these improve the applicability of fuzzy systems and can be adapted for any application.
2. Application-based: these are specific for particular applications and may not work effectively for other applications.

The MF represents all fuzziness encapsulated by a fuzzy set, hence formulating the MF is the essence of fuzzy logic operation. Intuition comprises semantic and contextual knowledge about the function under study; and also implicates linguistic knowledge. The essential factors include the estimated placement of the MF curve to the universe of discourse, the number of MFs to be used, and the overlapping of various MFs. The “rank ordering” evaluates preferences specified by an individual, a committee, a poll, or any other knowledge capturing methods and can be used to assign membership values to a fuzzy variable [18]. Some earlier methods for MF estimation relied on statistical methods rather than optimization. Some milestone literature regarding design problem employed in the proposed research are:

Devi and Sarma [24] proposed a technique to estimate MF form statistically defined probability density function (pdf) of data which is acquired from the histogram generated from a finite number of samples. An estimate of the probability mass function is depicted as:

\[ f(x) = \frac{1}{2N} \sum_{j=1}^{N} \varphi_{\Delta}(x_j) \tag{9} \]

Where,

\[ \varphi_{\Delta}(x_j) = \begin{cases} 
1, & x_j \in [x-h, x+h] \\
0, & \text{otherwise}
\end{cases} \tag{10} \]

The pdf can be approximated as:

\[ \hat{f}(x) = \frac{M_i(x)}{N_p(x)} = \frac{P(y)}{Q(y)} = \frac{p_0 + p_1 y^1 + \cdots + p_l y^l}{q_0 + q_1 y^1 + \cdots + q_p y^p} \tag{11} \]

Now, the fuzzy membership can be estimated as:

\[ \mu(x) = \frac{\hat{f}(x)}{\hat{f}(a)} \tag{12} \]

Where,
\[ \hat{f}(a) = \max \{ \hat{f}(x) \} \]  

(13)

In this method “arithmetic mean” is utilized to estimate the shape of the projected MF. Here “standard deviation” is used to determine the support for the FS. The experimental MFs thus obtained were utilized to identify Fisher Iris data [25] with a yield of 83.33% to 97.33% accuracy. This technique is an explicit inspiration for current research owing to direct utilization of standard deviation for estimating the support of the MF.

Civanlar and Trussell [20] proposed a method to obtain FSs using pdf and Zadeh’s possibility-probability consistency principle [26]. Civanlar formulated the consistency principle for real line sets as:

\[ \max_{x \in \mathcal{D}} \mu(x) = \int_{\mathcal{D}} p(x) \, dx \]  

(14)

Consistency principle postulates that the degree of possibility for any given event is either greater than or equal to the degree of probability, the resulting MF is utilized to satisfy the possibility probability principle. The necessity and sufficiency of the proposed criteria are yet to be found however the developed MF is found to be reasonable and the approach can be extended where so ever needed. The proposed method was utilized to eliminate noise from a set of linear equations. The method uses the pdf function generated from the statistical data. The optimal MF is depicted as:

\[ \mu(x) = \begin{cases} \alpha p(x) & \text{if } \alpha p(x) < 1 \\ 1 & \text{if } \alpha p(x) \geq 1 \end{cases} \]  

(15)

Where \( p(x) \) is pdf derived from the histogram of the used feature.

The optimal function can be found solving the below problem

\[ \text{minimize } f(\mu) = 1/2 \int_{-\infty}^{\infty} \mu^2(x) \, dx \]  

(16)

such that \[ G(\mu) = c - E[\mu] = c - \int_{-\infty}^{\infty} \mu(x)p(x) \, dx \leq 0, \]  

(17)

and \( \mu \in \Omega = \{ \mu(x), 0 \leq \mu(x) \leq 1 \}, c < 1 \)

The Lagrange for above function can be expressed as:

\[ L(\mu, \alpha) = 1/2 \int_{-\infty}^{\infty} \mu^2(x) \, dx + \alpha (c - \int_{-\infty}^{\infty} \mu(x)p(x) \, dx) \]  

(18)

Where \( \alpha \) is the Lagrange multiplier and \( \alpha \geq 0 \). Remark: this method does not employ any optimization technique nor is entropy calculated for the obtained FSs.

The FSs obtained through this method are determined by statistical “confidence levels” and can be depicted as:

![Figure 7. MFs obtained for different confidence levels [20]](image-url)

Dombi [22] compared various MFs based on their mathematical functions. These chosen MFs were based on various applications suggested by the author. The comparison was based on support for membership function \([a,b]\), the sharpness coefficient \( \lambda \), decision level \( \nu \). Dombi classified the MFs into following categories:

1. Heuristically based MFs.
2. MFs are based on reliability parameters.
3. MFs are based on theoretical decision making.
4. MFs are specifically utilized for designing control systems.
5. Linguistic models based MFs.

A thorough analysis of the MFs yields some common properties amongst various MFs:

1. All MFs were continuous and linear/piecewise linear (linearized if the intrinsic mathematical function is non-linear).
2. The MFs map a crisp interval of \([a,b]\) of a crisp set to an FS \(\mu[a,b] \rightarrow [0,1]\).
3. MFs are either monotonically increasing or decreasing or can be divided into both forms.

Combining all the MFs a new MF proposed by Dombi consists of two parts, a monotonically increasing:

\[
\mu(x) = \frac{(1 - \nu)^{\lambda-1}(x - a)^{\lambda} + \nu^{\lambda-1}(b - x)^{\lambda}}{1 - \nu^{\lambda-1}(x - a)^{\lambda} + \nu^{\lambda-1}(b - x)^{\lambda}}
\]  
(19)

And a monotonically decreasing:

\[
\mu(x) = \frac{(1 - \nu)^{\lambda-1}(b - x)^{\lambda} + \nu^{\lambda-1}(x - a)^{\lambda}}{1 - \nu^{\lambda-1}(b - x)^{\lambda} + \nu^{\lambda-1}(x - a)^{\lambda}}
\]  
(20)

Where \(x \in [a,b]\).

For the linear case, \(\lambda = 1\), we obtain:

\[
\mu(x) = \frac{x - a}{b - a}
\]  
(21)

Chen and Otto [27] proposed an interpolation based curve fitting method to estimate the MF from measurement data. The interpolation method employs Bernstein polynomial:

\[
B(g)(x) = \frac{g(x_i)(t - x_i)^2 + 2b(x - x_j)(t_i - x) + g(t_i)(x - t_i)^2}{(t_i - x_i)^2}
\]  
(22)

FSs are obtained to evaluate the maximum stress relations for computer-aided design engineering applications. Results are compared for (a) 3 points, (b) 6 points, and (c) 9 points measurement samples.

Hong, Lee [28] in their paper proposed an efficient way of learning through an algorithm based on training examples. This technique is used to develop an expert system. The system knowledge, pre-defined MF (triangular type) library are encoded in knowledge, and desired MF and decision rules are obtained using a learning algorithm. The first stage of the learning algorithm is searching for relevant attributes within the training data. Step-by-step procedure for this is as follows:

1. Sorting of the attribute (characteristics) values to ascending order.
2. Each attribute classifies the number of instances that belong to the same class.
3. Evaluate the fitness degree for each attribute, using:

\[
\frac{n - n_i}{n}
\]

where \(n\) is the total number of training instances and \(n_i\) is the \(i\)th training instance.
4. Now arrange the attributes in ascending order according to the fitness degree computed in the above steps.
5. Finally, select the relevant attribute.

The subsequent stage is the determination of MF. Step-by-step procedure for this is as follows:

1. Assign an initial default group to each selected attribute, \(G = 1 + 3 \log n\).
2. Determine the range for each of the attributes, \(R_i = \max(A_i) - \min(A_i)\).
3. Determine the group interval for each attribute, \(H_i = R_i / (G - 1)\).
4. Extend the possible minimum attribute value of the attribute \(A'_i\) to \(V_i = \min(A'_i) - H_i / 2\).
5. Divide the obtained range for each attribute into \(G\) groups.
6. Find the point \(b_i\) for each initial MF, \(b_i = \sum_{s=1}^{t_i} A'_i(t_i, s) / t_i\), where \(t_i\) is the \(s\)th interval for \(A'_i\), and \(t_i\) is the total number of instances in \(A'_i\).
7. Finally, determine the points \(a_i\) and \(c_i\) using:

\[
a_i = b_{i+1} \quad \text{and} \quad c_i = b_{i-1}.
\]

The proposed technique is employed to address the Fishers Iris data problem [25] and the results attained 100% accuracy for Setosa, 94% for Versicolour, and 92.72% for Virgininca, thus the experiment of the model gave a better performance and rational result. The overall classification efficiency achieved is 96.67%.

Tamaki et. al. [29] proposed a pdf based fuzzy observation constrained method to determine MF. For any function \(f(x)\) if pdf is determined by:

\[
P_i = \int_{-\infty}^{\infty} \chi_i(x) f(x) dx
\]  
(23)

Assuming a total of "t" FSs, \(S_1, S_2, \ldots, S_t\), the following constraint holds on:

\[
\forall x \in S_1 : S_{t+1}; \quad \chi_i(x) + \chi_{i+1}(x) = 1
\]  
(24)

In this method, the FS universe of discourse is partitioned into subsets consisting of "t" FSs. Required FSs are then selected from predefined sets based on their pdf. The MF selection is based on “discrimination degree”, and is defined as:

\[
\xi_{i-1} = \frac{\int_{\theta_i} f(x) dx}{P_{(t-1)a}}
\]  
(25)

where, \(\theta_i = \{x | \chi_{(i-1)b}\}\)

Hence the conditional probability of selecting \(S_i\) becomes:

\[
\chi_i(x) = \frac{P_{S_i chosen|x}}{P_{S_i chosen|y}}
\]  
(26)
Medasani et al. [30] stipulated an overview of numerous methods used for the determination of MF for pattern recognition applications. All the techniques for MF generation were covered which include: heuristics, histograms, probability to possibility transformations, nearest neighbor techniques, clustering, feed-forward neural networks, and mixture decomposition. Some of the MFs which are covered are triangular, piecewise linear, Gaussian, S-function, monotonic function, π-function, pdf based function, neural-network based methods, clustering based methods.

Concluding the comparison author states there is no distinctly best method, and every technique has its own merit depending on the problem. Medaglia et al [21] introduced a method based on Bézier curves [31] to fit an MF from given data points. This method allowed the designer to shape the MF based on the selection of control points. The smooth curve passing through the vicinity of set points is represented by mathematical formulation proposed by Bezier, and the MF of the proposed model is depicted as:

$$\mu_A(x(t)) = \begin{cases} 
0, & x(t) < m_l - \gamma, \\
\mu_{A_L}(x(t)), & m_l - \gamma \leq x(t) \leq m_l, \\
1, & m_l < x(t) < m_R, \\
\mu_{A_R}(x(t)), & m_R \leq x(t) \leq m_R + \beta, \\
0, & x(t) > m_R + \beta, 
\end{cases}$$

(27)

where \(\gamma\) and \(\beta\) are crossover points which are placed on the left side and right side of the MF respectively; \(m_L, m_R \in X\) are the lowest and highest full membership function values respectively. \(\mu_{A_L}(x(t))\) and \(\mu_{A_R}(x(t))\) are the left and right membership values. Further to compute the MF the collection of membership value required is calculated through a data-driven estimation method, in which the number of control points along with their location is determined. The sum of squared error between data and MF on the left and right-sided MF can be minimized using the following mathematical formulation.

$$\min_{i=2}^{M_R-1} \left( \sum_{k=0}^{n_R} \binom{n_R}{k} t_k^i (1-t_k)^{(n_R-k)} y_{R,k} \right)^2$$

(28)

Subject to: $$\sum_{k=0}^{n_R} \binom{n_R}{k} t_k^i (1-t_k)^{(n_R-k)} x_{R,k} = y_{R,i}$$

(29)

where MR is the number of data control points, \(y_{R,i}\) is the membership determined through direct approach, \(n_R\) is the degree of a polynomial on the right side. ML is the number of data points, \(y_{L,i}\) is the membership determined through direct approach, \(n_L\) is the degree of a polynomial on the left side. The proposed method can be used to estimate the MF for the imprecise concept. Through the properties of the Bezier curve, it was observed that the change is more significant near the control points and the MF changes completely. Error! Reference source not found. depicts the change in the MF when a single control point.

(a)
Figure 8. Effect of change in a single control point. (a) before change (b) after change [21].

In the area of pattern recognition as well the automatic generation of fuzzy MF is required. Yang, Bose [32] proposed a method of generating MF based on unsupervised learning through self-organizing feature map (SOFM) employing a feed-forward neural network. The robustness in performance for the system was tested on Fisher’s Iris data. MF using SOFM can be generated in two steps the first step is the clustering generation, to every feature input vector input weighted neurons are associated. the feature input vectors have a task of finding a winning neuron. The combination of two input features generates a new input feature to SOFM and SOFM becomes an MF generator. However in retrieving stage only one input feature is present, and after finding the winning neuron the output of SOFM is subvector of weight associated with labeling information, and the winning neuron will have information of both learning and retrieving phase and both the information can then be retrieved. The experiment carried out on the iris data sheet was found to be applicable for the multidimensional input feature with labeled information. The results showed MF to be consistent with pattern distribution in free space. In the proposed technique the neurons in the output layer of SOFM are limited but on the other hand number of rules has reduced. The location and form of the MF remain unaffected with the occurrence of elements. MF is found to be the function of the object and universe of discourse. McCloskey et. al. [33] proposed a dynamic method to obtain MFs using the data analysis technique. This method is proposed for financial planning and assists decision making for the corporate acquisition process. This method employs user scalable triangular MFs.

\[
\mu_A(x) = \begin{cases} 
0, & x \leq a \\
\frac{(x-a)}{(b-a)}, & a < x \leq b \\
\frac{(x-c)}{(b-c)}, & b < x < c \\
0, & x \geq c 
\end{cases}
\]  

(30)

Figure 9. Predefined MFs [33]

Error! Reference source not found. shows predefined MFs used for scaling purposes. Parameter \( b \) for MFs 1 and 5 is set to 0 and 1, respectively. And for MFs 2, 3, and 4 \( b \) is defined as:

\[
mf2(b) = \frac{\sum_{i=1}^{n}X_i}{2(n)}
\]  

(31)

\[
mf3(b) = \frac{\sum_{i=1}^{n}X_i}{n}
\]  

(32)
The proposed technique is used to predict acquisition decisions for 50 corporate entities. The results obtained exhibit 100% correspondence to the decision taken by the financial expert’s panel. The major limitation of this model is the requirement of a huge database for MF determination. This method however does not employ any optimization technique nor is entropy calculated for the obtained FSs.

Zhou, Khotanzad [34] proposed a genetic algorithm (GA) based method to determine the fuzzy MFs, size, and structure of rule base using training data. Consider the input vector as \( X \); and the number of MFs required is predetermined. Let the \( j \)th MF be a Gaussian type MF denoted by \( [\mu_j, \sigma_j] \). Let \( [R_{\text{mini}}, R_{\text{maxi}}] \) denote the minimum and maximum values of \( x_i \) and the training data be expanded by both sides by 20%. The real-valued \( \mu_j \) can be determined by:

\[
\mu_j = \mu_{\text{default}} + \frac{(M_j - 128)}{(128 \times \delta)}
\]

Where \( \delta = \frac{(R_{\text{maxi}} - R_{\text{mini}})}{(n + 1)} \) (34)

And \( \mu_{\text{default}} = R_{\text{mini}} + \delta \times \sigma_j, 1 < j \leq n \) (35)

Hence the possible range covered by \( x_i \) is divided equally among \( n \) MFs and then the location of the mean of each MF is rearranged depending on the computed \( M_j \) value. The conversion for variance is obtained through:

\[
\sigma_j = c_1 \times V_j + c_2
\]

Where \( c_1, c_2 \) are constants selected such that a \( V_j \) value of 64 covers 80% of the delta.

Larbes et. al. [35] proposed a GA optimization-based FLC for tracking MPPT in the photovoltaic system. Utilizing the model-free approach exhibited by FLC the designer has implemented FLC using the knowledge available from experts. FLC performance is further improved with the help of GA based online optimization. Initially normalized FSs are used which are equally divided throughout the universe of discourse. The designed FLC is then implemented for controlling the system. Online optimization is done to modify the predesigned MFs.

The objective function for online optimization is a minimization of ISE (Integral square error):

\[
J = \int e^2 dt
\]

Where, \( e = P_{\text{max}} - P \) (error). (Maximum power obtained from PV cell — desired power)

A major limitation of online optimization is the system should be operated to achieve the optimization results.

Nohe et. al. [36] proposed an optimization method based on GA for output regulation of a servomechanism with backlash.

The optimization was proposed for type-1 and type-2 FLS. Optimization methodology followed is:

1. Design the architecture of FLS: I/O selection, types of MFs to be used, initialize rule base.
Identify the parameters which are going to be varied for obtaining optimized MF.

Initialize the GA chromosome by sorting parameters of each variable respectively.

Define the objective function.

Select GA operations and desired optimization parameters.

Execute GA and replace the old MF parameters with optimized parameters.

The proposed research uses triangular MF for type-1 and type-2 FLS. The objective function for optimization is:

\[ \text{fitness}_i = \min(\text{mean}\{\text{error}\}) \] (39)

The results obtained indicate faster settling times and lower value of error indices as compared with non-optimized system performance. The major drawback of the proposed technique is the long execution time (350 hours) for achieving optimal results. Being an online optimization method a limitation exhibited by the system is that it must be operated to achieve the optimization results.

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Acilar and Arslan [37] proposed a clonal selection algorithm (CLONALG) for the optimization of support of a triangular MF. The CLONALG method is based on the immune system adaptability for various antigenic stimulus. The algorithm is inspired by the biological phenomena: only those cells which are adept at identifying an antigenic stimulus will thrive and differentiate into effectors cells. The primary features of the CLONALG are affinity proportional, reproduction, and mutation. The higher is the affinity of the function, the higher is the number of offspring generated. The mutation suffered by each immune cell during reproduction is inversely proportional to the affinity of the cell receptor with the antigen. It can be summarized as [38]:

1. Generate n antibodies randomly.
2. Repeat the following m times.
   a. Compute affinity (used as an objective function) of each antibody.
   b. Select j highest affinity antibodies.
   c. Selected j antibodies will be cloned as per there affinity function, thereby producing C clones.
   d. Clones are subjected to the hypermutation process, where higher the affinity lower is the hypermutation and vice versa.
   e. Compute affinity of clones C.
   f. From this generation of clones and antibodies, select n highest affinity cells to generate a new population.
   g. Replace the l lowest affinity population by the new generation generated randomly.
3. Repeat till m.

The affinity function defined by the author is:

\[ \text{Affinity} = \maxerror - \text{totalerror} \] (38)

Here,

\[ \text{totalerror} = \sum_{i=1}^{n} (y_{\text{CLONALG}}^{i} - y_{i})^{2} \] (39)

\[ \maxerror = \sum_{i=1}^{n} (y_{i} - y_{\text{max}})^{2} \] (40)

Where, \( y_{i} \) is the \( i \)th reference i/p of o/p. \( y_{\text{CLONALG}}^{i} \) is obtained by CLONALG is output for \( i \)th reference i/p of o/p and \( n \) is the number of i/o data.

The performance of the proposed optimization method is compared to GA and BPSO algorithms. Results indicate faster convergence for CLONALG as compared to GA or BPSO (binary particle swarm optimization). Being an online optimization method a limitation exhibited is that the system should be operated to achieve the optimization results.

Ang and Quek [39] proposed a bio-inspired algorithm-based method for the generation of MFS. Supervised Pseudo Self-Evolving Cerebellar (SPSEC) algorithm draws its inspiration from the development of the human nervous system. Where the initial architecture is laid out without any activity-dependent processes and is continuously refined from activity-dependent ways. The SPSEC algorithm is used to generate MF by reconciling with the semantics of the human interpretable linguistic terms. The generated MFs do not deviate
from human-interpretable linguistic variables thereby eliminating the requirement of further optimization [40]. The authors tested the MF generated using this method for conformity of decision boundaries and results indicate. The decision boundaries for the MF generated through the proposed technique were found to be consistent with the inherent probabilistic decision boundaries corresponding to the training data set.

Kao et. al. [41] proposed a hybrid density-based clustering method for MF generation in a fuzzy controller designed for a ball mill pulverizing system. The proposed technique is a two-stage MF generation process where it initially uses nearest neighbor similarity and is based on fuzzy weighted distance considered as the distance metric between two objects. Following which it utilizes, a density-based spatial clustering of applications with noise (DBSCAN) inspired clustering process where the divisive clustering and agglomerative methods are used to detect the clusters. The obtained clusters are then projected into the domains of dimensions to determine the MFs. The fuzzy controller developed from the generated MF is used for real-time control of the ball-pulverizing process and the experiment results verify the proficiency of the proposed technique.

IV. OPTIMIZATION OF MF USING FUZZY ENTROPY

Shannon laid the concepts of communication systems by formulating the fundamentals for information theory [42] [43]. The uncertainty associated with random variables is measured by entropy function or simply “entropy determines the expected value of information available in a corresponding message”. However, the entropy of a fuzzy variable is defined as a measure of the fuzziness in a given FS or system as a degree of variable randomness, & uncertainty associated is used as an information measure. FLS are amongst systems that employ Shannon’s findings to fuzzy logic-based clustering, decision making, optimization techniques to name a few. Fuzzy entropy can be defined as the measure of fuzzy information available obtained through a FS.

\[ H(A) = -\int_{-\infty}^{\infty} \mu_i \log \mu_i + (1 - \mu_i) \log (1 - \mu_i) \]

Here \( \mu_i \) is membership value of element \( i \) in a FS \( A \). This expression is also written as:

\[ H(A) = \int_{-\infty}^{\infty} f(\mu(x))dx \]

Here \( f(x) = -x \ln x - (1 - x) \ln (1 - x) \)

Cheng and Chen [44] used the notion of maximum entropy for finding the MF. The FS defined represents the membership value for the brightness of gray level in a given image. The brightness function is given by:

\[ \text{bright} = \sum_{r_k \in \Omega} \mu_{\text{bright}}(r_k) / r_k \]

The MF \( \mu_{\text{bright}} \) expresses the brightness of a gray level \( r_k \). \( \Omega = \{ r_0, r_1, ..., r_l - 1 \} \).

The probability of this fuzzy event is depicted as:

\[ P(\text{bright}) = \sum_{k=0}^{l-1} \mu_{\text{bright}}(r_k) P(r_k) \]

Here \( P \) is the probability measure of the occurrence of gray level in an image. The MF of the brightness of gray level is represented by an s-function:

\[ S(x) = \begin{cases} 0, & x \leq a, \\ \frac{(x - a)^2}{(b - a)(c - a)}, & a \leq x \leq b, \\ \frac{1 - (c - b)(c - a)}{(x - c)^2}, & b \leq x \leq c, \\ 1, & x \geq c, \end{cases} \]

Where \( x \) is an independent variable \( a, b, c \) are parameters with \( b \) as a center of the MF \( a \) and \( c \) be the crossover points. The range of the fuzzy event is decided by parameter \( a \) and \( c \), parameter \( b \) defines the increasing rate, which widely affects the shape of the s-function. As the brightness of gray levels for any image is contextual dependent, obtaining the optimized value of parameters is often a challenging job. Choosing s-function as a MF the parameters \( a, b, \) and \( c \) defining the brightness can be formulated as:

\[ \left( \text{bright}; a_{\text{opt}}, b_{\text{opt}}, c_{\text{opt}} \right) = \max \{ H(\text{bright}; a, b, c) | r_0 \leq a < b < c \leq r_{l-1} \} \]

To obtain the optimized solution for brightness function formulated in equation (46) simulated annealing algorithm is used. Experimental results yield a rapid decrease in the number of iterations required to reach the optimum result. Simulated annealing algorithm converged to a solution in about 300 iterations, which is 1/900th number of iterations undertook by the exhaustive search algorithm.

Hasuik et al [45] proposed an optimization method to integrate fuzzy entropy and pdf to obtain optimal FSs. The performance of the proposed technique was compared with the modified S-curve based MF
optimization method. However, due to the complexity of the optimization indices, the proposed algorithm was found to be computationally intensive. The method for constructing MF based on fuzzy Shannon entropy and human’s interval estimation integrating pdf with fuzzy Shannon entropy. Proposed optimization is based on a modified s-function formulated by Peidro and Vasant [46]. The modified s-curve functions are based on human cognitive behavior thereby creates a MF which fits close to human subjectivity. The left-hand membership function is depicted in Error! Reference source not found. for which the mathematical equation can be written as:

\[
S_l(x) = \begin{cases} 
1 & x^b < x \\
\frac{1 - \frac{B_1}{1 + C_1 e^{\alpha_1 x}}}{w^b}, & x^a < x < x^b \\
w^a, & x^a = x \\
0 & x < x^a
\end{cases} \quad (47)
\]

Here \( B_1, C_1, \) and \( \alpha_1 > 0 \) are the parameters that determine the shape of the s-curve. \( w^a, w^b \) are the constants whose values are close to 1 and 0 and are set by the expert.

Similarly, the right-hand of the s-curve membership function is depicted by:

\[
S_r(x) = \begin{cases} 
1, & x < x^c \\
w^c, & x^c = x \\
\frac{B_2}{1 + C_2 e^{\alpha_2 x}}, & x^c < x < x^d \\
w^d, & x^d = x \\
0, & x^d < x
\end{cases} \quad (48)
\]

The objective of the method is to maximize the fuzzy entropy under s-curve function, where pdf is given as a constraint. The objective function can be defined as:

\[
\text{maximize} = -\int_{-\infty}^{\infty} \{\mu(x) \log_2(\mu(x)) + (1 - \mu(x)) \log_2(1 - \mu(x))\} \, dx \\
\text{subject to} \quad E(\mu(x)) = \int_{-\infty}^{\infty} \mu(x)p(x) \, dx \geq m, \quad 0 \leq \mu(x) \leq 1, \forall x
\]

Where \( m \) is the average membership value initially determined and \( p(x) \) is pdf.

Optimization examples include uniformly distributed pdfs and partially uniform distributed pdfs generating appropriate MFs. However a disadvantage of the proposed method is large convergence time for obtaining optimization results, also for complicated pdfs the solution is hard to obtain. To overcome the computational challenges Hasuike, et al [47] proposed an algorithm based on piecewise linear functions utilizing Lagrange functions. Using the least-squares method equation (49) can be rewritten as:

\[
\text{Minimize} \quad \sum_{i=1}^{n+1} \{\mu_i \log_2 (\mu_i) + (1 - \mu_i) \log_2 (1 - \mu_i)\} + W \sum_{i=2}^{n} (\mu_{i+1} - 2\mu_i + \mu_{i-1})^2 \\
\text{subject to} \quad \sum_{i=1}^{n+1} \mu_i p(x_i) \geq c, \quad 0 \leq \mu_i \leq 1, \quad (i = 1, 2, \ldots, n)
\]

Here \( W \) is the weight for smoothing of the membership function. The Lagrange function for the above problem is given by

\[
L = \sum_{i=1}^{n+1} \{\mu_i \log_2 (\mu_i) + (1 - \mu_i) \log_2 (1 - \mu_i)\} + W \sum_{i=2}^{n} (\mu_{i+1} - 2\mu_i + \mu_{i-1})^2 + \lambda \left( c - \sum_{i=1}^{n+1} \mu_i p(x_i) \right) \\
+ \sum_{i=2}^{n-1} \mu_i (\mu_i - 1) - \sum_{i=2}^{n-1} \delta_i \mu_i
\]

The smoothing function further minimize the objective function as

\[
\text{Minimize} \quad \sum_{i=1}^{n+1} \{\mu_i \log_2 (\mu_i) + (1 - \mu_i) \log_2 (1 - \mu_i)\} + W \sum_{i=1}^{n} (\mu_i - (i - 1)/n)^2 \\
\text{subject to} \quad \sum_{i=1}^{n+1} \mu_i p(x_i) \geq c, \quad 0 \leq \mu_i \leq 1, \quad (i = 1, 2, \ldots, n + 1)
\]

The optimization of above mentioned objective function is carried out using mathematical programming. The smoothing function however reduces the computational requirement of the optimization problem, but it is achieved at the cost of loss in generality of the objective function.
V. ASSESSMENT OF MEMBERSHIP FUNCTION FORMULATION TECHNIQUES

The objective of this paper is to discuss various membership function formulation techniques along with their applications. Table 1 summarizes these techniques and provides concluding remarks for them.

Table 1. Summary and research gap

<table>
<thead>
<tr>
<th>Reference</th>
<th>Description</th>
<th>Key findings</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>B Bharathi Devi et al [5]</td>
<td>Pattern recognition. Probability density function is obtained from the histogram of a finite number of samples. Results for Iris flower data set recognition are discussed.</td>
<td>Mean determines the shape of the estimated membership function. Standard deviation determines the support of the membership function. Results obtained indicated an accuracy of 83.33% to 97.33%.</td>
<td>No entropy computation. No optimization technique. Explicit inspiration for current research.</td>
</tr>
<tr>
<td>Chuen Chien Lee [4]</td>
<td>Formalization of systematic approach for FLC. Survey paper with focus on formalizing FLC.</td>
<td>Step by step approach to build a fuzzy logic based controller.</td>
<td>First formal paper to conceptualize and formulate fuzzy logic control theory.</td>
</tr>
<tr>
<td>Futoshi Tamaki et. al [10]</td>
<td>Crisp sets are converted to triangular fuzzy sets using discrimination degree.</td>
<td>Probability density function based support determination. Discrimination degree is obtained using standard normal distribution.</td>
<td>No entropy computation. No optimization technique. Explicit inspiration for current research.</td>
</tr>
<tr>
<td>Anthony McCloskey et. al[12]</td>
<td>User scalable triangular fuzzy MF for corporate and financial planning. Based on data acquisition through various experts.</td>
<td>Support for MF is dynamically computed according to the data analysis. Result for 50 corporate entities were computed and exhibited a</td>
<td>No entropy computation. No optimization technique. Explicit inspiration for current research.</td>
</tr>
</tbody>
</table>
VI. CONCLUSION

The paper is intended to help the researchers looking to optimally utilize the fuzzy logic for engineering applications. Various membership function formulation techniques along with their working algorithms, formulation have been discussed in sections 3 & 4. For researcher working in the core field of fuzzy engineering section 5 summarizes the key developments in the membership function estimation. The description column provides the key principle of membership estimation by the respective author. The key findings column gives detail to the problem where the authors have implemented the algorithm and the results obtained from the proposed method. The remark column describes the research gap which can be addressed by future researchers. This summarization can be utilized to identify promising methods and gaps in those which can be further explored to improve the membership function formulation further. An important aspect of membership function formulation which is generally overlooked is estimation of support for fuzzy sets [48], which marks further research scope in the field.

VII. REFERENCES

2009.


